Pre-Calculus

Name: Key

Notes—(6.2) Dot Product of Vectors

Objectives: You will be able to calculate dot products, the angle between two vectors, and projections of vectors.

DEFINITION --- Dot product

The ***dot product*** or ***inner product*** of $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ is → $\mathbf{u} \bullet \mathbf{v} = u_1v_1 + u_2v_2$

$i = \langle 1, 0 \rangle$

$j = \langle 0, 1 \rangle$

Ex1) Find each dot product:

a) $\langle 3, 4 \rangle \bullet \langle 5, 2 \rangle$

$3 \cdot 5 + 4 \cdot 2$

$15 + 8$

$\boxed{23}$

b) $\langle 1, -2 \rangle \bullet \langle -4, 3 \rangle$

$1 \cdot -4 + -2 \cdot 3$

$-4 + -6$

$\boxed{-10}$

c) $(2\mathbf{i} - \mathbf{j}) \bullet (3\mathbf{i} - 5\mathbf{j})$

$2 \cdot 3 + -1 \cdot -5$

$6 + 5$

$\boxed{11}$

Properties of the Dot product -------- Let **u**, **v**, and **w** be vectors and let c be a scalar.

1. $\mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u}$
2. $\mathbf{u} \bullet \mathbf{u} = |\mathbf{u}|^2$
3. $\mathbf{0} \bullet \mathbf{u} = 0$
4. $\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w}$
   $(\mathbf{u} + \mathbf{v}) \bullet \mathbf{w} = \mathbf{u} \bullet \mathbf{w} + \mathbf{v} \bullet \mathbf{w}$
5. $(c\mathbf{u}) \bullet \mathbf{v} = \mathbf{u} \bullet (c\mathbf{v}) = c(\mathbf{u} \bullet \mathbf{v})$

Ex2) Use the dot product to find the length of vector $\mathbf{u} = \langle 4, -3 \rangle$

magnitude

$u \cdot u = \|u\|^2$

$\langle 4, -3 \rangle \cdot \langle 4, -3 \rangle =$

$4 \cdot 4 + -3 \cdot -3$

$\sqrt{25} = \sqrt{\|u\|^2}$

$\|u\| = 5$

THEOREM -------- **Angle Between Two Vectors** $\cos\theta = \dfrac{u \cdot v}{|u||v|}$ and $\theta = \cos^{-1}\left(\dfrac{u \cdot v}{|u||v|}\right)$

Ex3) Find the angle between two vectors **u** & **v**.

no calc.

a) $\mathbf{u} = \langle 2, 3 \rangle$, $\mathbf{v} = \langle -2, 5 \rangle$

$u \cdot v$

$2 \cdot -2 + 3 \cdot 5$

$-4 + 15$

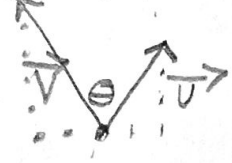
$\boxed{11}$

$\|u\| = \sqrt{2^2 + 3^2} = \sqrt{13}$

$\|v\| = \sqrt{2^2 + 5^2} = \sqrt{29}$

$\cos\theta = \dfrac{11}{\sqrt{13}\sqrt{29}}$

$\theta \approx 55.49°$



b) $\mathbf{u} = \langle 2, 1 \rangle$, $\mathbf{v} = \langle -1, -3 \rangle$

$u \cdot v$

$2 \cdot -1 + 1 \cdot -3$

$-2 + -3$

-5

$\|u\| = \sqrt{2^2 + 1^2} = \sqrt{5}$

$\|v\| = \sqrt{(-1)^2 + (-3)^2} = \sqrt{10}$

$\cos\theta = \dfrac{-5}{\sqrt{5} \cdot \sqrt{10}}$

$\cos\theta = \dfrac{-5}{5\sqrt{2}}$

$\cos\theta = \dfrac{-1}{\sqrt{2}}$

$\theta = \dfrac{3\pi}{4}$ or $135°$

(Perpendicular)

Definition ---------- Orthogonal Vectors → The vectors **u** and **v** are orthogonal if and only if $\mathbf{u} \cdot \mathbf{v} = 0$

Ex4) Prove that the vectors $\mathbf{u} = \langle 2, 3 \rangle$ and $\mathbf{v} = \langle -6, 4 \rangle$ are orthogonal.

$2 \cdot -6 + 3 \cdot 4$

$-12 + 12$

0

Projection of u onto v ------- If u and v are nonzero vectors, the projection of u onto v is $\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$

Ex5) Find the vector projection of $\mathbf{u} = \langle 6, 2 \rangle$ onto $\mathbf{v} = \langle 5, -5 \rangle$. Then write **u** as the sum of two orthogonal vectors, one of which is $\text{proj}_{\mathbf{v}}\mathbf{u}$.

NOW YOU TRY ☺

6) Find the vector projection of $\mathbf{u} = \langle -3, 4 \rangle$ onto $\mathbf{v} = \langle 12, -5 \rangle$. Then write **u** as the sum of two orthogonal vectors, one of which is $\text{proj}_{\mathbf{v}}\mathbf{u}$.